The Masterball Puzzle

David Joyner and Andrew Southern

11-10-96

Contents

1 Introduction

To begin, a description of the masterball. There are several patterns - a beachball striped pattern (marketed as the "geomaster"), a dragon pattern ("dragonmaster"), and several others. They are mechanically identical but the dragonmaster, for example, has a greater number of visible combinations than the geomaster. We shall focus on the geomaster but our moves are useful for the other masterballs as well. From now on, "masterball" means the beachball patterned masterball.

A masterball sphere has 32 tiles of 8 distinct colors. We shall assume that the masterball is in a fixed position in space. A geodesic path from the north pole to the south pole is called a **longitudinal line** and a closed geodesic path parallel to the equator is called a **latitudinal line**. There are 8 longitudinal lines and 3 latitudinal lines on the masterball. In spherical coordinates, the longitudinal lines are at $\theta = n\pi/4$, $n = 1, .., 8$, and the latitudinal lines are at $\phi = \pi/4, \pi/2, 3\pi/4$. The sphere shall be oriented by the right-hand-rule - the thumb of the right hand wrapping along the polar axis points towards the north pole. We assume that one of the longitudinal lines has been fixed once and for all. This fixed line shall be labeled "1", the next line (with respect to the orientation above) as "2", and so on.

Allowed manuevers: One may rotate the masterball east-to-west by multiples of $\pi/4$ along each of the 4 latitudinal bands or by multiples of π along each of the 8 longitudinal lines. The set of all possible finite sequences of such manuevers forms a group, called the **masterball group**. There are $4 + 8 = 12$ generators of this group (this set is not minimal). The group will be described a little more later.

A **facet** will be one of the 32 subdivisions of the masterball created by these geodesics. A facet shall be regarded as immobile positions on the sphere and labeled either by an integer $i \in \{1, ..., 32\}$ or by a pair $(i, j) \in [1, 4] \times [1, 8]$, whichever is more convenient at the time. If a facet has either the north pole or the south pole as a vertex then we call it a **small** (or **polar**) facet. Otherwise, we call a facet **large** (or **middle** or **equatorial**). A **coloring** of the masterball will be a labeling of each facet by one of the 8 colors in such a way that

(a) each of the 8 colors occurs exactly twice in the set of the 16 small facets,

(b) each of the 8 colors occurs exactly twice in the set of the 16 large facets.

Remark 1 *Every geomaster we've seen has "Masterball (c)" imprinted on a yellow small facet. This distinguishing feature shall be ignored, though it could be used as a convenient reference facet.*

A **move** of the masterball will be a change in the coloring of the master-ball associated to a sequence of manuevers as described above.

If we now identify each of the 8 colors with an integer in $1, ..., 8$ and identify the collection of facets of the masterball with a 4×8 array of integers in this range. To "solve" an array one must, by an appropriate sequence of moves corresponding to the above described rotations of the masterball, put this array into a "rainbow" position so that the matrix entries of each column has the same number. Thus the array or "colors"

$$\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8
\end{array}$$

or, in matrix notation,

$$\begin{array}{cccccccc}
11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\
21 & 22 & 23 & 24 & 25 & 26 & 27 & 28 \\
31 & 32 & 33 & 34 & 35 & 36 & 37 & 38 \\
41 & 42 & 43 & 44 & 45 & 46 & 47 & 48
\end{array}$$

is "solved" and corresponds to the picture below.

The array of colors

$$
\begin{array}{cccccccc}
6 & 7 & 8 & 1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8
\end{array}
$$

corresponds to the picture below.

2 Notation

The generators for the latitudinal rotations are denoted r_1, r_2, r_3, r_4, so for example,

$$
r_1 :
\begin{matrix}
11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 \\
21 & 22 & 23 & 24 & 25 & 26 & 27 & 28 \\
31 & 32 & 33 & 34 & 35 & 36 & 37 & 38 \\
41 & 42 & 43 & 44 & 45 & 46 & 47 & 48
\end{matrix}
\longmapsto
\begin{matrix}
12 & 13 & 14 & 15 & 16 & 17 & 18 & 11 \\
21 & 22 & 23 & 24 & 25 & 26 & 27 & 28 \\
31 & 32 & 33 & 34 & 35 & 36 & 37 & 38 \\
41 & 42 & 43 & 44 & 45 & 46 & 47 & 48
\end{matrix}
,
$$

pictured as follows:

Using cycle notation, we may write r_1 as

$$(11\ 12\ 13\ 14\ 15\ 16\ 17\ 18).$$

As you look down at the ball from the north pole, this move rotates the ball *clockwise*. The other moves r_2, r_3, r_4 are defined similarly - they also rotate the associated band clockwise as viewed from the north pole.

2.1 Extrinsic notation

The generators for the longitudinal rotations are denoted $f_1, f_2, ..., f_8$, so for example,

$$
f_1 : \begin{array}{cccccccc}
12 & 13 & 14 & 15 & 16 & 17 & 18 & 11 \\
21 & 22 & 23 & 24 & 25 & 26 & 27 & 28 \\
31 & 32 & 33 & 34 & 35 & 36 & 37 & 38 \\
41 & 42 & 43 & 44 & 45 & 46 & 47 & 48
\end{array}
\longmapsto
\begin{array}{cccccccc}
44 & 43 & 42 & 41 & 16 & 17 & 18 & 11 \\
34 & 33 & 32 & 31 & 25 & 26 & 27 & 28 \\
24 & 23 & 22 & 21 & 35 & 36 & 37 & 38 \\
15 & 14 & 13 & 12 & 45 & 46 & 47 & 48
\end{array}.
$$

From a solved masterball, f_1 is pictured as follows:

With these rules, one can check the relation

$$f_5 = r_1^4 * r_2^4 * r_3^4 * r_4^4 * f_1 * r_1^4 * r_2^4 * r_3^4 * r_4^4.$$

There are similar identities for f_6, f_7, f_8. Also, one can check that

$$r_1 = (f_3 * f_7)^{-1} * r_4^{-1} * f_3 * f_7.$$

There are similar identities for r_2, r_3, r_4.

2.2 Intrinsic notation

Another system notation is sometimes more convenient.

Fix once and for all a reference facet somewhere on the masterball. In practice you may want to keep a thumb on this facet so we shall call this the **thumb facet** or the **marked piece**. The the longitudinal lines and the corresponding 180° longitudinal rotations moves shall be described relative to this thumb facet, where ever it may have moved to. There is a longitudinal

line immediately to the "east" of the thumb facet. This fixed line shall be labeled "1", the next line (with respect to the orientation mentioned earlier) as "2", and so on. The generators for the longitudinal rotations are denoted s_1, s_2, s_3, s_4. We do not use the moves $s_5, ..., s_8$. The notation for the r_1, r_2, r_3, r_4 is the same as in the extrinsic notation.

3   The number of possible positions

Identifying the facets of the masterball with the entries of the array

$$\begin{array}{cccccccc}
8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\
16 & 15 & 14 & 13 & 12 & 11 & 10 & 9 \\
24 & 23 & 22 & 21 & 20 & 19 & 18 & 17 \\
32 & 31 & 30 & 29 & 28 & 27 & 26 & 25
\end{array}\; ,$$

we may express the generators of the masterball group in disjoint cycle notation as a subgroup of S_{32} (the symmetric group on 32 letters):

$$
\begin{aligned}
r_1 &= (1,2,3,4,5,6,7,8), \\
r_2 &= (9,10,11,12,13,14,15,16), \\
r_3 &= (17,18,19,20,21,22,23,24), \\
r_4 &= (25,26,27,28,29,30,31,32), \\
f_1 &= (5,32)(6,31)(7,30)(8,29)(13,24)(14,23)(15,22)(16,21), \\
f_2 &= (4,31)(5,30)(6,29)(7,28)(12,23)(13,22)(14,21)(15,20), \\
f_3 &= (3,30)(4,29)(5,28)(6,27)(11,22)(12,21)(13,20)(14,19), \\
f_4 &= (2,29)(3,28)(4,27)(5,26)(10,21)(11,22)(12,23)(13,24), \\
f_5 &= (1,28)(2,27)(3,26)(4,25)(9,20)(10,19)(11,18)(12,17), \\
f_6 &= (8,27)(1,26)(2,25)(3,32)(16,19)(9,18)(10,17)(11,24). \\
f_7 &= (7,26)(8,25)(1,32)(2,31)(15,18)(16,17)(9,24)(10,23), \\
f_8 &= (6,25)(7,32)(8,31)(1,30)(14,17)(15,24)(16,23)(9,22),
\end{aligned}
$$

These generate a subgroup G (denoted `perm_ball` in **rainbow**) of S_{32} of order $N = 43776313669739505254 4000000$.

In fact, the elements $f_1, f_2, f_3, f_4, r_1, r_2, r_3, r_4$ generate this group. According to GAP's `AbStab.g` share package, the move f_1 is not necessary, so G is generated by at most 7 elements. The derived subgroup of this group is

8

of index 4 and after that the decreasing chain of derived subgroups stabilizes, according to MAPLE calculations.

The group G acts on the set of all possible colorings. The masterball itself contains several symmetries which prevent this group from acting freely. For example, the red "north pole facet" can be switched with the red "south pole facet" without changing the masterball's position. We may also rotate the entire ball "east-to-west" by $\pi/4$ radians without affecting the masterball's position. As another example, we can simply turn the ball over, swapping the north pole with the south pole. There are a total of $n_1 n_2 n_3 n_4 = 2^{20}$ such symmetries, where n_1 denotes the number of north-south pole swaps (2), n_2 denotes the number of latitudinal rotations by multiples of $\pi/4$ (8), n_3 denotes the number of like-colored "north-south pole facet" swaps (2^8), and n_4 denotes the number of like-colored "upper middle-lower middle facet" swaps (2^8). Moreover, to solve the masterball you need only achieve a beach-ball pattern. One a solved masterball, you can swap the colors around in $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$ ways and get another solved masterball. Accounting for these "symmetries", we conclude that the number of positions of the masterball is therefore $N/(2^{20} \cdot 7!) \cong 8.2 \times 10^{17}$. This is about 50 times smaller than the number of positions of the 3×3 Rubik's cube (see, for example, §2.4 in [B], or [S]).

3.1   Heuristics on God's algorithm

This subsection presents some heuristics regarding God's algorithm for the masterball.

Let G denote the rainbow masterball group. The is the group generated by the basic moves $r_1, ..., r_4, f_1, ..., f_8$ but we identify two moves if they are the same under the symmetries above. Each move $g \in G$ may be written in the form

$$g = g1 * g_2 * ... * g_n$$

where either

$$g_i = \begin{cases} r_1^{a_{1i}} * r_2^{a_{2i}} * r_3^{a_{3i}} * r_4^{a_{4i}}, & i \text{ odd, some } a_{ki} \in \{0, ..., 7\}, \\ \prod_i f_{k_i}^{b_i}, & i \text{ even, some } b_i \in \{0, 1\}, \end{cases}$$

or

$$g_i = \begin{cases} r_1^{a_{1i}} * r_2^{a_{2i}} * r_3^{a_{3i}} * r_4^{a_{4i}}, & i \text{ even, some } a_{ki} \in \{0, ..., 7\}, \\ \prod_i f_{k_i}^{b_i}, & i \text{ odd, some } b_i \in \{0, 1\}. \end{cases}$$

Assume for the moment, heuristically, that there are at least $4^4\dot{4} = 1024$ different g_i and that there are no relations or cancellations between the different terms in the product. Since the number of different positions is about 8×10^{17}, so the maximal number s of g_i in a solution is bounded by the condition $1024^s < 8 \times 10^{17}$, so $s < 6.9$. If each g_i accounts for, say, m moves, our assumption implies that there are at most $6m$ moves in God's algorithm. If g_i is a product of the $r'_j s$ then $m \leq 32$. If g_i is a product of the $f'_j s$ then m is the size of God's algorithm for the group $F = < f_1, f_2, f_3, f_4 >$.

4 Strategies

4.1 The latitudinal strategy

One strategy, developed by the second author, is the following:

- solve the two equatorial regions (the ones moved by r_2 and r_3) relative to each other,

- solve the upper polar region (the one moved by r_1) relative to the equatorial regions, using "polarswap" moves (given below),

- finish off the lower use a "polarswap" move to solve the polar region (the one moved by r_4) relative to the equatorial regions, using "polarswap" moves (given in the "catalog" below).

We call this the **latitudinal strategy**.

4.2 The longitudinal strategy

The first few steps below describes the idea of "fishing". This will solve a longitudinal half of the masterball. The remaining steps use "polarswaps" and "equatorswaps" (given in the "catalog" below).

1. The idea is to first get all the middle bands aligned first, so you get ball corresponding to a matrix of the form

$$
\begin{array}{cccccccc}
* & * & * & * & * & * & * & * \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
* & * & * & * & * & * & * & *
\end{array}
$$

Here, * denotes any color. We have labeled the colors on the masterball as 1, 2, ..., 8 in order of occurrence.

(Mathematically, this amounts to performing some carefully choosen commutators.) Without too much trouble you can always assume that we have one column aligned. You may need to flip or rotate the ball a little bit to do this. Call this aligned column "column 1" and call the color in column 1, "color 1". We want to get the middle two entries in column 2 aligned. Call the color in the $(2,3)$-entry "color 2".

We want to get color 2 in the $(2,2)$-entry. The remaining large color 2 tile is the "crab" we will "fish" for. Hold the ball in front of you in such a way that column 2 is slightly to the left of center and column 3 is slightly to the right of center. There are 4 facets in the right upper middle band, 4 facets in the left upper middle band, 4 facets in the right lower middle band and 4 facets in the left lower middle band. A flip about the center on the right half (i.e., perform f_2) exchanges these. We may assume that color 2 is on one of the four facets in the right lower middle band. (If it isn't you need to apply f_2 first). Now perform $r_2^{-1}f_2^{-1}r_2f_2$: first perform r_2^{-1} (this is "baiting the hook"), then f_2^{-1} ("putting the hook in the water"), then r_2 ("setting the hook"), and finally f_2 ("reeling in the hook"). You may or may not have color 2 in the $(2,2)$ place like you want but the color 1 stripe is intact. If necessary, try again. After at most 4 tries you'll be successful.

2. Repeat this "fishing" strategy to get color 2 in the $(1,2)$ position (using $r_1^{-1}f_2^{-1}r_1f_2$ in place of $r_2^{-1}f_2^{-1}r_2f_2$). Now, by turning the ball over if necessary, repeat this idea to get color 2 in the $(4,2)$ position. Now you have two "aligned" stripes on your ball - color 1 in column 1 and color 2 in column 2. We say, in this case, that columns 1 and 2 have been "solved".

3. Repeat this for columns 3 and 4.

4. Assume that we have four columns completely aligned: columns 1, 2, 3 and 4. Place the ball so that columns 1, 2 are slightly to the left of center and columns 3, 4 are slightly to the right. Perform a flip about the middle (i.e., an f_3). Now rotate the entire ball by 45° so that two solid stripes are "left-most" (say columns 1,2) and two solid stripes are "right-most" (say columns 3,4). The ball looks more symmetric about

the center this way. The flip about the middle will be simply f_4 in this set up. Now try using the following types of manuevers: ("middle switching"): f_4, f_8, $r_2 f_4 r_2^{-1} f_4^{-1} r_2^{-1} f_4 r_2 f_4^{-1}$, $r_3 f_4 r_3^{-1} f_4^{-1} r_3^{-1} f_4 r_3 f_4^{-1}$, $r_2 f_8 r_2^{-1} f_8^{-1} r_2^{-1} f_8 r_2 f_8^{-1}$, $r_3 f_8 r_3^{-1} f_8^{-1} r_3^{-1} f_8 r_3 f_8^{-1}$. These may be performed in any order. The point is that after each such manuever, the 4 solid columns will remain solid, whereas the "mixed up" columns may remain mixed up.

5. Use polarswaps and equatorswaps, given in the next section, to solve the remainder of the masterball.

4.3 The "north-south" strategy

Another strategy developed by the second author is the following:

- solve most of the north equatorial regions (the ones moved by r_2) with with north polar regions (moved by r_1) relative to each other,

- solve most of the the south equatorial regions (the ones moved by r_3) with with south polar regions (moved by r_4) relative to each other,

- match up the northern regions with their southern counterparts using "fishing" moves,

- finish off using "polarswap" and "equatorswap" moves (given in the "catalog" below).

We call this the **north-south strategy**.

5 Catalog of useful moves

5.1 Column moves

We number the columns as $1, ..., 8$. We will use a signed cycle notation to denote an action of a move on the columns of the masterball. For example,

(a) a move which switches the $1st$ and $3rd$ column but flips both of them over will be denoted by $(1, 3)_-$,

(b) a move which sends the $4th$ column to the $6th$ column, the $6th$ column to the $5th$ column, and switches the $2nd$ and $3rd$ column but flips both of them over will be denoted by $(2, 3)_-(6, 5, 4)$.

move	cycle
f_1	$(1,4)_-(2,3)_-$
f_2	$(2,5)_-(3,4)_-$
f_3	$(3,6)_-(4,5)_-$
f_4	$(4,7)_-(5,6)_-$
f_5	$(5,8)_-(6,7)_-$
f_6	$(1,6)_-(7,8)_-$
f_7	$(2,7)_-(1,8)_-$
f_8	$(3,8)_-(1,2)_-$
$f_1 * f_2 * f_1$	$(1,2)_-(3,5)$
$f_1 * f_2 * f_1 * f_2$	$(5,4,3,2,1)_-$
$f_1 * f_3 * f_1$	$(1,5)(2,6)$
$f_2 * f_3 * f_2$	$(2,3)_-(6,5,4)$
$f_1 * f_4 * f_1$	$(1,7)(5,6)$
$f_1 * f_5 * f_1$	$(5,8)_-(6,7)_-$
$f_1 * f_8 * f_1$	$(2,8)(3,4)_-$
$f_8 * f_1 * f_8$	$(1,8)_-(4,3,2)$
$f_2 * f_1 * f_2$	$(1,3)(4,5)_-$
$f_3 * f_1 * f_3$	$(1,3)(4,8)_-$
$f_8 * f_1 * f_2$	$(1,4)_-(2,3,8,5)_-$

Finally, $(f_1 * f_2 * f_3 * f_4)^2 * r_1 * r_2 * r_3 * r_4$ swaps the 7,8 columns and leaves all the others fixed but flipped over.

Remark 2 *We call a move of the masterball* **column preserving** *if it sends columns to other columns (possibly flipping north-to-south as well). For example, any f_i is column preserving, but so is $r_1 * r_2 * r_3 * r_4$. Each column preserving move may be thought of as a permutation of the eight columns. Because of the last mentioned move, one can show that any permutation of the columns can be achieved.*

5.2 Some products of 2-cycles on the facets

These are all based on an idea of the second-named author. The polar2swap36 and equator2swap36 were obtained help of a MAPLE implementation of the masterball.

We number the facets in the i-th column, north-to-south, as $i1, i2, i3, i4$ (where $i = 1, 2, ..., 8$).

move	cycle
$x = r_1 * f_4 * r_1^{-1} * r_4 * f_4 * r_4^{-1}$	$(41,\ 84)(44,\ 81)$
$x * r_1^4 * x * r_4^4$	$(41,\ 81)(44,\ 84)$
$f_1 * r_1 * f_4 * r_1^{-1} * r_4 * f_4 * r_4^{-1} * f_1$	$(14,\ 84)(11,\ 81)$
polar2swap36	$(11,\ 14)(31,\ 61)$
polar2swap18	$(61,\ 64)(11,\ 81)$
equator2swap36	$(12,\ 13)(32,\ 62)$
equator2swap18	$(62,\ 63)(12,\ 82)$

where

$$\text{polar2swap36} =$$
$$f_1 * r_3^{-1} * r_4^{-1} * f_1 * f_2 * r_1 * r_4^{-1} * f_2 * r_4^4 * f_2 * r_1^{-1} * r_4 * f_2 * r_4^4 * f_1 * r_3 * r_4 * f_1$$

(note, if you replace r_3 by r_2 both times in this move you get the same effect),

$$\text{polar2swap18} =$$
$$f_1 * r_3^{-1} * r_4^{-1} * f_3 * f_4 * r_1 * r_4^{-1} * f_4 * r_4^4 * f_4 * r_1^{-1} * r_4 * f_4 * r_4^4 * f_3 * r_3 * r_4 * f_1$$

$$\text{equator2swap36} =$$
$$f_1 * r_4^{-1} * r_3^{-1} * f_1 * f_2 * r_2 * r_3^{-1} * f_2 * r_3^4 * f_2 * r_2^{-1} * r_3 * f_2 * r_3^4 * f_1 * r_4 * r_3 * f_1$$

and

$$\text{equator2swap18} =$$
$$f_1 * r_4^{-1} * r_3^{-1} * f_3 * f_4 * r_2 * r_3^{-1} * f_4 * r_3^4 * f_4 * r_2^{-1} * r_3 * f_4 * r_3^4 * f_3 * r_4 * r_3 * f_1$$

Using these moves, along with "fishing" to solve half the masterball, one can solve the rainbow masterball. As an exercise, we leave it to the reader to translate these moves into the intrinsic notation.

5.3   Some 2-cycles

In this subsection, we use list notation for the moves, so they may be directly plugged into the MAPLE rainbow "rb" command (described below) if desired.

move	cycle
NSmidswap	$(52, 53)$
NSswap	$(51, 54)$
EWswap	$(27, 28)$

where

```
NSmid_swap:=[r3,r2,f1,r2,f1,r3,f1,r2^(-1),f1,r3^(-2),
f1,r2,f1,r3,f1,r2^(-1),f1,r3^4,f1,
r2,f1,r3,f1,r2^(-1),f1,r3^(-2),f1,r2,f1,
r3^2,f1,r2^(-1),f1,r3^2,f1,r2,
f1,r3^(-1),f1,r2^(-1),f1,r3^(-1),f1,r2,f1,
r3,f1,r2^(-1),f1,r2,r3^(-2),f1,r2^(-1),f1,
r3^(-1),f1,r2,f1,r2^(-1),r3,f1,r2,f1,
r2^(-1),f1,r2,f1,r3^2,f1,
r2^(-1),f1,r2,f1,r2^(-1),f1,r2,r3^(-1),
f1,r2^(-1),f1,r3,f1,r2,f1,r2^(-2),r3^2];
```

(replacing r_2, r_3 with r_1, r_4 gives)

```
NS_swap:=[r4,r1,f1,r1,f1,r4,f1,r1^(-1),f1,r4^(-2),
f1,r1,f1,r4,f1,r1^(-1),f1,r4^4,f1,
r1,f1,r4,f1,r1^(-1),f1,r4^(-2),f1,r1,f1,
r4^2,f1,r1^(-1),f1,r4^2,f1,r1,
f1,r4^(-1),f1,r1^(-1),f1,r4^(-1),f1,r1,f1,
r4,f1,r1^(-1),f1,r1,r4^(-2),f1,r1^(-1),f1,
r4^(-1),f1,r1,f1,r1^(-1),r4,f1,r1,f1,
r1^(-1),f1,r1,f1,r4^2,f1,
r1^(-1),f1,r1,f1,r1^(-1),f1,r1,r4^(-1),
f1,r1^(-1),f1,r4,f1,r1,f1,r1^(-2),r4^2];
```

and

```
EW_swap:=[f2,f3,f5,r2,f5,r1,r2,r3,
r4,f5,r2,f5,r3^(-1),f5,r2^(-1),f5,r3^(-1),f5,
r2,f5,r3^(-1),f5,r2^(-1),f5,r3^3,f5,
r2,f5,r3^(-1),f5,r2^(-1),f5,r3^2,f5,r2,
f5,r3^(-1),f5,r2^(-1),f5,r3^2,r2,f5,
r2^(-1),f5,r2,f4,r4^(-1),r1^(-1),f5,r3^(-1),f5,r2^(-1),f5,
r3^(-1),f5,r2,f5,r3,f5,r2^(-1),f5,r3^2,
f5,r2,f5,r3^(-1),f5,r2^(-1),f5,r3^(-1),f5,r2,
f5,r3,f5,r2^(-1),f5,r2,f5,r2^(-1),f5,
r3^2,r2^2,r3^(-1),f5,r2,f5,r3^2,f5,r2^(-1),
f5,r3^(-3),f5,r2,f5,r3^2,f5,
r2^(-1),f5,r3^(-1),f5,r2^(-1),f5,r2,f5,r2^(-1),f5,
```

```
r2,r3^(-1),f5,r2^(-1),f5,r3^(-1),f5,r2,f5,
r3,f5,r2^(-1),f5,r3,f5,r2,f5,r2^(-1),
f5,r2,f5,r2^(-2),r3,f5,r2,f5,
r2^(-2),f5,r2^(-1),f5,f3,f2];
```

These moves require considerable patience and the ability to make moves without looking at the masterball.

5.4 A pretty pattern

The following patterns are due to the second-named author.

The "quadrantized pattern" is obtained from the beachball pattern using the move

$$(r_3 * r_4)^{-4} * f_1 * (r_3 * r_4)^2 * f_2 * (r_3 * r_4)^2 * f_3,$$

pictured as follows:

Some "checker patterns" are obtained from the beachball pattern using the move

$$c_i = r_1 * r_3 * f_i * r_3^{-1} * r_1^{-1} * f_i, \quad i = 1, 2, 3, 4.$$

c_1 will "checker" the 1,5 columns (it also swaps and flips the 2,4 columns and leaves all facets on other columns alone). By "checker" we mean that the south pole facet of column 1 goes to the north pole facet of column 5 and the north-middle facet of column 1 goes to the south-middle facet of column 5. The other facets of columns 1,5 stay the same. Now try c_1, c_2, c_3, c_4, perhaps in combination with some appropriate "flips" f_j to achieve some pretty checkered patterns.

6 The computer as a scratchpad

This section will discuss the use of some files, available from the first author, for MAPLE and GAP. If you don't have these, skip this section. The MAPLE program below does not solve the masterball, it only simulates it on the computer - a "virtual" masterball if you will. However, the GAP commands described below can be used (with some work) to solve the masterball. The solution is not very efficient in general. It appears to be possible, with a lot of work, to link the GAP and MAPLE programs together in Windows 95 to create a single program which solves the masterball but we have not done so.

Incidently, there are versions of the MAPLE program described below for the 3×3 Rubik's cube, the $4\times$ Rubik's cube, the skewb, the magic jewel (a Rubik's octahedron), and the equator puzzle (with no orientations on the pieces). The GAP program describd below also works for them. See the www page

        http://www.nadn.navy.mil/MathDept/wdj/homepage.html

6.1 MAPLE commands

First, open a MAPLEV4 worksheet and load the packages linalg and plots. If you want to use the `master_squares` or `square_draw` commands, also load the geom package. If you want to use any of the group theory commands also load the group package.

Copy the rainbow.txt file into a directory (eg, `c:\maplev4\share\games\rainbow` if you are using a Windows machine) and the read it into your maple session (by typing
`read('c:/maplev4/share/games/rainbow/rainbow.txt');`
for example).

6.1.1 To play the game (in Maple)

I shall now assume that you have a MasterBall in front of you. The initial masterball position will be described as a 4×8 matrix as above. Call this matrix $A0$. Now pick a word in the group, enter say
```
w:=[r1,f1,r3^3];
```
and type
```
rb(w,A0);
```
to "rotate" the ball (hence the "rb") according to the group element w. You try to choose manuevers which eventually will put the sphere into "rainbow" form.

6.2 The solution using GAP

This idea seems to work in principal but in practice is can lead to solutions which are too long to be practical.

To solve the rainbow masterball, assume that you have represented the initial position as an 4×8 matrix $A0$.

You need to determine the element of the masterball group $G \subset S_{32}$ corresponding to this position. Call this element $L0$, written as a list of lists in MAPLE's disjoint cycle notation.

The following question naturally arises: Can we find (or at least get MAPLE or GAP to find) an expression for $L0$ as a product of the generators $r_1, ..., f_8$ of G? This is a special instance of the well-known "word problem" in group theory.

I shall now assume that you have the GAP 3.4 package (GAP is a freeware package produced by Lehrstuhl D für Mathematik, RWTH Aachen) and the GAP share package AbStab.g (loaded as abstab.g in the directory `c:\gap\gap3r4p3\lib\abstab.g` if you have a dos machine, otherwise you will need to modify the rainbow.g file included in an appendix below).

After starting gap, log your session into some text file, rainbow.log say, by typing
```
LogTo("rainbow.log");
```
Now load the rainbow.g file by typing
```
    Read("c:/gap/rainbow/rainbow.g");
```
assuming that your path to the file is `c:\gap\rainbow` (otherwise, modify the path accordingly). Enter the permutation element describing the masterball position into GAP. Call it w. (The GAP notation for a permutation

18

is not the same as the MAPLE notation but they are similar. There are some MAPLE procedures included in rainbow.txt to help you describe this permutation but they, unfortunately, do not always return disjoint cycles.) The command

```
w in G;
```

will return "true" if w is an element of the masterball group G. The AbStab command

```
soln:=FactorPermGroupElement(G, w );
```

will return the solution of the masterball as a word in the generators $g1, g2, ...$ choosen by AbStab. This can take some time and yield a fairly long expression, depending on w. The AbStab command

```
soln_short:=Shrink(G, w );
```

will try to produce a shorter solution of the masterball as a word in the generators $g1, g2, ...$ choosen by AbStab.

These generators are a subset of the generators $r1, ..., f8$. You need to find out which is which. For example, to find which (if any) generator $r1$ is, type

```
FactorPermGroupElement(G, r1 );
```

For example, you might find (as I did in one session)

```
g1=r4, g2=r3, g3=r2, g4=f5, g5=f4, g6=f3, g7=f2, g8=r1.
```

In any case, you need to know these $gi's$. Now the permutation `soln` is in GAP notation. By exiting GAP and loading rainbow.log into an ASCII text editor you can modify `soln` into MAPLE notation (replace all the $gi's$ by the appropriate ri or fi, replace '*' by ',', replace `^-1` by `^(-1)` for example, and save the resulting expression as a list by putting a [in front and a]; in back). This can now be pasted into MAPLE and the masterball solved using the `rb` command as discussed in the previous subsection.

Remark 3 *What is remarkable is that, at least in principle, we have an "expert system" (one which can solve the masterball) with no built-in "knowledge" of the masterball other than its group structure.*

7 Other sources of information

Some more information on the masterball as well as pictures of many of the moves discussed above can be found on the www page

    http://www.nadn.navy.mil/MathDept/wdj/mball/rainbow.html

References

[B]   C. Bandelow, <u>Inside Rubik's cube and beyond</u>, Birkhaüser Boston, 1980

[S]   D. Singmaster, <u>Notes on the Rubik's cube</u>, Enslow Press, 1981